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On the Degree of Approximation by Extended Hermite–Fejér Operators

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1. INTRODUCTION

The Hermite–Fejér interpolatory polynomial, $H_n(f; x)$, of degree $\leqslant 2n - 1$ is defined by

$$H_n(f; x) = \sum_{k=1}^{n} f(x_{kn})(1 - xx_{kn})(T_n(x)/n(x - x_{kn}))^2$$

where $x_{kn} = \cos((2k - 1)/2n)\,\pi$, $k = 1, 2,..., n$, are the zeros of the Tchebychev polynomial $T_n(x) = \cos(n \arccos x)$. Bojanic [1] has shown that, for any continuous function f defined on $[-1, 1]$,

$$\| H_n(f; \cdot) - f \| \leqslant (C/n) \sum_{k=1}^{n} \omega(f, 1/k), \tag{1.1}$$

where $\|f\| = \sup\{|f(x)|: -1 \leqslant x \leqslant 1\}$, C is a constant, and $\omega(f, \delta)$ is the modulus of continuity of f on $[-1, 1]$.

Let $\{\alpha_n\}$ be a sequence of positive numbers strictly increasing to ∞. For any real runction $f(t)$ defined on $(-\infty, \infty)$, the extended Hermite–Fejér operators are defined by

$$H_n(f(\alpha_n t); x/\alpha_n) = \frac{1}{n^2} \sum_{k=1}^{n} f(\alpha_n x_{kn}) \left(1 - \frac{xx_{kn}}{\alpha_n}\right)\left(\frac{T_n(x/\alpha_n)}{x/\alpha_n - x_{kn}}\right)^2.$$

Hsu [4] has shown that these extended Hermite–Fejér operators may be used to approximate continuous functions, $f(x)$, of any order of growth as $|x| \to \infty$. In this note we investigate the degree of approximation by the extended Hermite–Fejér operators.

2. Preliminaries

In the sequel let $e_k(x) = x^k$ for $k = 0, 1, 2, \ldots$.

LEMMA 2.1. *Let $\{L_n\}$ be a sequence of linear operators which are positive (see [2] or [3]) on $[-c, c]$, $0 < c < \infty$, with common domain D, and let $\{\alpha_n\}$ be a sequence of positive numbers strictly increasing to ∞. Let $-\infty < a < x < b < \infty$, $f \in D \cap C(-\infty, \infty)$, $e_i \in D$ $(i = 0, 1, 2)$, and $L_n(e_0; x) \equiv 1$. If there exists a number $p > 1$ and a positive increasing function Ω such that $\Omega^p \in D$ and $f(t) = O(\Omega(|t|))(|t| \to \infty)$, then, for $n \geqslant M = M(x)$,*

$$|f(x) - L_n(f(\alpha_n t); x/\alpha_n)| \leqslant 2\omega(f, p_n) + m_x^{-2} |f(x)| p_n^2$$
$$+ C_1[L_n((\Omega(\alpha_n |t|))^p; x/\alpha_n)]^{1/p} m_x^{-2/p'} p_n^{2/p'},$$

where

$$(1/p) + (1/p') = 1, \qquad m_x = \min\{|a - x|, |b - x|\},$$
$$\omega(f; [a, b]; \delta) = \max\{|f(\mu) - f(\nu)| : |\mu - \nu| \leqslant \delta, \mu, \nu \in [a, b]\},$$
$$\psi_{nx}(t) = (\alpha_n t - x)^2, \qquad p_n = p_n(x) = [L_n(\psi_{nx}; x/\alpha_n)]^{1/2},$$

and C_1 is a constant depending only on f such that $|f(t)| \leqslant C_1\Omega(|t|)$, $-\infty < t < \infty$.

Lemma 2.1 is the easy extension of [3, Theorem 2.2] to the whole real line.

LEMMA 2.2. *Let $\{\alpha_n\}$ be an increasing sequence of positive numbers. Let Ω and its derivative Ω' be positive increasing functions on $(0, \infty)$. Then*

$$\left| H_n\left(\Omega(\alpha_n |\cdot|); \frac{x}{\alpha_n}\right) - \Omega(|x|) \right| \leqslant 2C\left(\frac{1 + \log n}{n}\right) \alpha_n^2 \Omega'(\alpha_n),$$

where C is Bojanic's constant from (1.1).

If $p > 1$, we have

$$\left| H_n\left(\Omega^p(\alpha_n |\cdot|); \frac{x}{\alpha_n}\right) - \Omega^p(|x|) \right| \leqslant 2Cp\left(\frac{1 + \log n}{n}\right) \alpha_n^2 \Omega^{p-1}(\alpha_n)\, \Omega'(\alpha_n).$$

Proof. Let $g(x) = \Omega(\alpha_n \mid x \mid)$ for $x \in [-1, 1]$. Then by (1.1), for every $x \in [-\alpha_n, \alpha_n]$,

$$\begin{aligned}| H_n(\Omega(\alpha_n \mid \cdot \mid); x/\alpha_n) - \Omega(\mid x \mid)| &\leqslant \| H_n(\Omega(\alpha_n \mid \cdot \mid)) - \Omega(\alpha_n \mid \cdot \mid)\|_{[-1,1]} \\ &\leqslant (C/n) \sum_{k=1}^{n} \omega(g, 1/k)\end{aligned}$$

and the proof is completed by showing that

$$\omega(g, h) \leqslant 2h\alpha_n^2\Omega'(\alpha_n).$$

For example, if $| x | \leqslant h$, $| y | \leqslant h$ and $| x - y | \leqslant h < 1$, we have

$$\begin{aligned}| g(x) - g(y)| &\leqslant \Omega(\alpha_n \mid x \mid) - \Omega(0) + \Omega(\alpha_n \mid y \mid) - \Omega(0) \\ &\leqslant 2(\Omega(\alpha_n h) - \Omega(0)) \\ &\leqslant 2h\alpha_n^2\Omega'(\alpha_n).\end{aligned}$$

If $| h | < x \leqslant 1$ we have $y = y - x + x > h - | x - y | > 0$ and so

$$\begin{aligned}| g(x) - g(y)| &= | \Omega(\alpha_n x) - \Omega(\alpha_n y)| \\ &\leqslant \alpha_n^2 \mid x - y \mid \Omega'(\alpha_n) \\ &\leqslant h\alpha_n^2\Omega'(\alpha_n),\end{aligned}$$

and the remainder follows easily.

3. Main Results

THEOREM 3.1. *Let $\{\alpha_n\}$ be a sequence of positive numbers strictly increasing to ∞ and $-\infty < a < x < b < \infty$. Let $f \in C(-\infty, \infty)$ and suppose there exist a number $p > 1$ and positive increasing functions Ω, Ω' such that*

$$f(x) = O(\Omega(\mid x \mid))(\mid x \mid \to \infty).$$

Choose $N = N(x)$ such that $x \in [-\alpha_N, \alpha_N]$. If $n \geqslant N(x)$, then

$$\begin{aligned}| H_n(f(\alpha_n t); x/\alpha_n) - f(x)| &\leqslant 2\omega(f, \alpha_n(2/n)^{1/2}) + m_x^{-2} \mid f(x)\mid 2\alpha_n^2 n^{-1} \\ &+ C_1[\Omega^p(\mid x \mid) + 2Cp((1 + \log n)/n)\, \alpha_n^2\Omega^{p-1}(\alpha_n)\, \Omega'(\alpha_n)]^{1/p} \\ &\times m_x^{-2/p'}(2\alpha_n^2 n^{-1})^{1/p'}\end{aligned}$$

where C is Bojanic's constant, and $p, p', m_x, \omega(f; [a, b]; \delta)$ and C_1 are as in Lemma 2.1.

Proof. First, if $\psi_{nx}(t) = (\alpha_n t - x)^2$,

$$H_n(\psi_{nx}(t); x/\alpha_n) = \sum_{k=1}^{n} (\alpha_n x_{kn} - x)^2 A_{kn}(x/\alpha_n)$$

$$= \left(\frac{\alpha_n T_n(x/\alpha_n)}{n}\right)^2 \sum_{k=1}^{n} \left(\frac{x}{\alpha_n} - x_{kn}\right)^2 \cdot \left(1 - \frac{x x_{kn}}{\alpha_n}\right) \Big/ \left(\frac{x}{\alpha_n} - x_{kn}\right)^2.$$

If $n \geqslant N(x)$ and $x \in [-\alpha_N, \alpha_N]$, then

$$-1 \leqslant x/\alpha_n \leqslant 1 \qquad \text{and} \qquad 0 \leqslant 1 - x_{kn}(x/\alpha_n) \leqslant 2.$$

Thus

$$p_n(x) = [H_n(\psi_{nx}(t); x/\alpha_n)]^{1/2} \leqslant \alpha_n(2/n)^{1/2}.$$

Now it is easy to see that $H_n(e_0; x) \equiv 1$ and the Hermite–Fejér operators are positive on $[-1, 1]$. The result now follows from Lemmas 2.1 and 2.2. Denote

$$\exp_{m+1}(x) = \exp(\exp_m x) \qquad \text{and} \qquad \log_{m+1}(x) = \log(\log_m x).$$

Theorem 3.1 has an immediate application to a result of Hsu [4]:

THEOREM 3.2. *For any continuous function $f(x)$ defined on $(-\infty, \infty)$ and satisfying the order condition $f(x) = O(\exp_{m+1} |x|)(|x| \to \infty)$,*

$$\lim_{n\to\infty} H_n(f(t \log_{m+2}(n)); x/\log_{m+2}(n)) = f(x)$$

almost uniformly on $(-\infty, \infty)$.

The rate of convergence in Theorem 3.2 is given by choosing $\alpha_n = \log_{m+2}(n)$ in Theorem 3.1.

We call Ω a modulus of continuity on $[0, \infty)$ if Ω is continuous and non-decreasing on $[0, \infty)$; $\Omega(t_1 + t_2) \leqslant \Omega(t_1) + \Omega(t_2)$ for $t_1, t_2 > 0$; $\Omega(\lambda t) \leqslant (\lambda + 1)\,\Omega(t)$ for $\lambda, t > 0$; $\Omega(0) = 0$; and $(1/t_2)\,\Omega(t_2) \leqslant (2/t_1)\,\Omega(t_1)$ if $t_1 < t_2$. If Ω is a modulus of continuity and M is a positive constant, let $C_M(\Omega) = \{f \mid f \text{ is continuous on } (-\infty, \infty) \text{ and } |f(x) - f(t)| \leqslant M\Omega(|x - t|)$ for all $x, t\}$. Let $a > 0$, $\{\alpha_n\}$ be a positive sequence strictly increasing to ∞, and $\| g \| = \sup\{| g(t)|: -a \leqslant t \leqslant a\}$. Define

$$E(H_n; C_M(\Omega); \alpha_n; a) = \sup\{\| H_n(f(\alpha_n t); x/\alpha_n) - f(x)\| : f \in C_M(\Omega)\}.$$

The next result is an extension of Bojanic's result [1] to the approximation of functions unbounded on the real line.

THEOREM 3.3. *There exist constants K_1, $K_2 > 0$ such that for any modulus of continuity Ω on* $[0, \infty)$ *and positive sequence* $\{\alpha_n\}$ *strictly increasing to* ∞, *and all* $n \geqslant N(a, \alpha_n)$

$$K_1 \frac{M}{n} \sum_{k=1}^{n} \Omega\left(\frac{\alpha_n}{k}\right) \leqslant E(H_n ; C_M(\Omega); \alpha_n ; a) \leqslant K_2 \frac{M}{n} \sum_{k=1}^{n} \Omega\left(\frac{\alpha_n}{k}\right).$$

The proof of Theorem 3.3 follows the lines of [2, Theorem 7.11, pp. 232–237], once we note the following lemma (compare with [2, Lemma 7.1, p. 228].

LEMMA 3.4. *Let Ω be a modulus of continuity on* $[0, \infty)$ *and* $\{\alpha_n\}$ *be a positive sequence strictly increasing to* ∞. *If* $n \geqslant 2$, *then*

$$(\pi\alpha_n/n) \int_{\pi\alpha_n/n}^{\pi\alpha_n} t^{-2}\Omega(t)\, dt \leqslant \sum_{k=1}^{n-1} k^{-2}\Omega(\alpha_n n^{-1}(k+1)\,\pi)$$

$$\leqslant (8\pi\alpha_n/n) \int_{\pi\alpha_n/n}^{\pi\alpha_n} t^{-2}\Omega(t)\, dt.$$

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REFERENCES

1. R. BOJANIC, "A Note on the Precision of Interpolation by Hermite–Fejér Polynomials,", pp. 69–76, Proceedings of the Conference on the Constructive Theory of Functions, Budapest, 1969.
2. R. A. DEVORE, "The Approximation of Continuous Functions by Positive Linear Operators, Lecture Notes in Mathematics, Springer–Verlag, New York, 1972.
3. S. EISENBERG AND B. WOOD, On the order of approximation of unbounded functions by positive linear operators, *SIAM J. Numer. Anal.* **9** (1972), 266–276.
4. C. HSU, On a kind of extended Fejér–Hermite interpolation polynomials, *Acta Math. Acad. Sci. Hungar.* **15** (1964), 325–328.